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On the Group Theory of 2-Dimensional Noncommutative Quantum Mechanics

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ABSTRACT

We examine here, in some detail, the symmetry groups of a non-relativistic quantum system of 2 spatial degrees of freedom. It is argued that the (2+1) centrally extended Galilei group $G_{\text{Gal}}^{\text{ext}}$ can be considered as one of the underlying symmetry groups from the algebraic point of view. Later, the phase space variables of a particle moving on a 2 dimensional plane are quantized using the coherent states arising from the unitary dual of $G_{\text{Gal}}^{\text{ext}}$. The resulting quantized variables are found to satisfy the set of commutation relations which are postulated in the existing literatures of noncommutative quantum mechanics. We also consider a further extension resulting in the noncommutativity between the underlying momentum coordinates and explore how this is connected to the triple central extension of the abelian group of translations in $\mathbb{R}^4$ and its unitary irreducible representations.

1. Introduction

In this paper, we focus on the quantum mechanical features of a non-relativistic system of 2 degrees of freedom at very short distances. The failure of standard quantum mechanics at such distances has been extensively discussed in the current literature (see, for example, Delduc *et al.* (2008) and

Scholtz *et al.* (2009)). It has been argued that at distances as small as the Planck length, the position observables along different coordinate axes fail to commute. This is in addition to the canonical commutation relations (CCR) of standard quantum mechanics, so that one writes

$$[Q_i, P_j] = i\hbar\delta_{ij}I, \;\; i,j = 1,2, \qquad [Q_1, Q_2] = i\vartheta I, \tag{1}$$

Here Q_i and P_j are the position and momentum coordinates, respectively. I is the identity operator on the underlying Hilbert space and ϑ is a small positive parameter that controls the additional noncommutativity between the position coordinates. In the limit $\vartheta = 0$, Q_i and P_j behave as the usual quantum mechanical position and momentum operators.

Now one could have also started with the following set of noncanonical commutation relations

$$\begin{aligned}[Q_i, P_j] &= i\hbar\delta_{ij}I, \;\; i,j = 1,2, \qquad [Q_i, Q_j] = i\vartheta\epsilon_{ij}I, \\ &[P_i, P_j] = i\gamma\epsilon_{ij}I, \;\; i,j = 1,2\ ,\end{aligned}$$

where ϵ_{ij} is the totally antisymmetric tensor and γ being yet another positive parameter. Physically, γ insinuates the presence of a constant magnetic field in the system.

The organization of the paper is as follows. First we ask if Q_i and P_j appearing in (1) and (2) really represent the generators of some Lie group. If so, then what are the possible symmetry groups of NCQM? Then we extract the relevant coherent states from the appropriate unitary irreducible representations of the underlying group and subsequently carry out coherent state quantization of the phase space variables of the system moving in a 2 dimensional plane. The quantized variables are found to satisfy the above set of commutation relations.

2. The role of (2+1) Galilei group in NCQM of 2-plane

The (2+1) Galilei group G_{Gal} is a 6 parameter Lie group. It is the kinematical group of a classical, non-relativistic space-time of (2+1) dimension. The group parameters are the time translations b, space translations $\mathbf{a}$, rotations R of the two-dimensional space and velocity boosts $\mathbf{v}$. The action of G_{Gal} on a (2+1) space-time point $(\mathbf{x}, t)$ is given by

$$\mathbf{x} \longrightarrow \mathbf{x}' = R\mathbf{x} + \mathbf{a} + \mathbf{v}t, \quad t \longrightarrow t' = t + b, \tag{2}$$

with the group composition law

$$(R, b, \mathbf{v}, \mathbf{a})(R', b', \mathbf{v}', \mathbf{a}') = (RR', b + b', \mathbf{v} + R\mathbf{v}', \mathbf{a} + R\mathbf{a}' + \mathbf{v}b'). \qquad (3)$$

The Lie algebra $\mathfrak{G}_{\text{Gal}}$ of G_{Gal} has a 3 dimensional vector space of central extensions. The extended Lie algebra now reads (see, for example, Bose (1995a) and Bose (1995b))

$$\begin{aligned} [M, N_i] &= \epsilon_{ij} N_j & [M, P_i] &= \epsilon_{ij} P_j \\ [H, P_i] &= 0 & [M, H] &= \mathfrak{h} \\ [N_i, N_j] &= \epsilon_{ij} \mathfrak{d} & [P_i, P_j] &= 0 \\ [N_i, P_j] &= \delta_{ij} \mathfrak{m} & [N_i, H] &= P_i, \end{aligned} \qquad (4)$$

where we denote the three generators of the center as $\mathfrak{h}, \mathfrak{d}$ and $\mathfrak{m}$. Also, P_i, N_i, M and H generate the space translations, boosts along the two spatial directions, angular momentum and time translation, respectively. Since G_{Gal} is not simply connected, not all three central extensions survive their passage from the algebra to the group level (see Bose (1995a)). The second cohomology group for G_{Gal} turns out simply to be 2 dimensional with $\mathfrak{h}$ being identically zero in (4). Let us denote this double central extension of the (2+1)-Galilei group as $G^{\text{ext}}_{\text{Gal}}$ and the corresponding Lie algebra as $\mathfrak{G}^{\text{ext}}_{\text{Gal}}$.

We denote a generic group element of $G^{\text{ext}}_{\text{Gal}}$ as $(\theta, \phi, R, b, \mathbf{v}, \mathbf{a}) = (\theta, \phi, r)$, where r stands for an element of G_{Gal}. The meanings of $R, b, \mathbf{v}$ and $\mathbf{a}$ have already been put forth earlier in this section and the two additional group parameters introduce the 2 fold central extension carried out. The two cocycles pertaining to the central extension read off as

$$\begin{aligned} \xi^1_m(r;\, r') &= e^{\frac{im}{2}(\mathbf{a}\cdot R\mathbf{v}' - \mathbf{v}\cdot R\mathbf{a}' + b'\mathbf{v}\cdot R\mathbf{v}')}, \\ \xi^2_\lambda(r;\, r') &= e^{\frac{i\lambda}{2}(\mathbf{v}\wedge R\mathbf{v}')}\,, \quad \text{where} \quad \mathbf{q}\wedge\mathbf{p} = q_1 p_2 - q_2 p_1\,, \end{aligned} \qquad (5)$$

where we have adopted the vector notation given by $\mathbf{q} = (q_1, q_2),\ \mathbf{p} = (p_1, p_2)$. The group composition rule for $G^{\text{ext}}_{\text{Gal}}$ then reads

$$\begin{aligned} &(\theta, \phi, R, b, \mathbf{v}, \mathbf{a})(\theta', \phi', R', b', \mathbf{v}', \mathbf{a}') \\ &\quad = (\theta + \theta' + \xi^1_m(r;\, r'),\ \phi + \phi' + \xi^2_\lambda(r;\, r'), \\ &\qquad RR', b + b', \mathbf{v} + R\mathbf{v}', \mathbf{a} + R\mathbf{a}' + \mathbf{v}b')\,. \end{aligned} \qquad (6)$$

The unitary irreducible representations of $G^{\text{ext}}_{\text{Gal}}$, realized on $L^2(\mathbb{R}^2, d\mathbf{k})$, are all characterized by an ordered pair (m, ϑ) of reals and s that is measured in half integral multiples of $\hbar$. Physically, m represents the mass of the nonrelativistic system under study and ϑ is some other exotic parameter, the physical significance of which will follow in due course. The parameter ϑ, making its appearance in (1) and (2), is actually expressible in terms of the mass m and λ given by (5). We shall consider only the case $m \neq 0$ and $\lambda \neq 0$ in this paper. The quantity s is to be interpreted as the eigenvalue of the intrinsic angular momentum operator S. The relevant unitary irreducible representations are all computed in (Bose (1995b)).

In view of the above mentioned facts, the basis of the Lie algebra $\mathfrak{G}_{\text{Gal}}^{\text{ext}}$ can be represented as appropriate self-adjoint operators on $L^2(\mathbb{R}^2, d\mathbf{k})$, obeying the following set of commutation rules.

$$\begin{aligned}
[\hat{M}, \hat{N}_i] &= i\epsilon_{ij}\hat{N}_j \qquad & [\hat{M}, \hat{P}_i] &= i\epsilon_{ij}\hat{P}_j \\
[\hat{H}, \hat{P}_i] &= 0 \qquad & [\hat{M}, \hat{H}] &= 0 \\
[\hat{N}_i, \hat{N}_j] &= i\epsilon_{ij}\lambda\hat{I} \qquad & [\hat{P}_i, \hat{P}_j] &= 0 \\
[\hat{N}_i, \hat{P}_j] &= i\delta_{ij}m\hat{I} \qquad & [\hat{N}_i, \hat{H}] &= i\hat{P}_i \; .
\end{aligned} \tag{7}$$

The self-adjoint operators $\hat{N}_i$, $\hat{M}$, $\hat{P}_i$ and $\hat{H}$ represent the group generators N_i, M, P_i and H, respectively. It is noteworthy in this context that both the central generators $\mathfrak{d}$ and $\mathfrak{m}$ of $G_{\text{Gal}}^{\text{ext}}$ are represented by the identity operator $\hat{I}$ of the underlying Hilbert space $L^2(\mathbb{R}^2, d\mathbf{k})$.

Consider now the 2-dimensional *nononcommutative Weyl-Heisenberg group*, of which the group generators Q_i, P_j, and I are subject to the commutation relations (1). The associated Lie algebra is also referred to as the *noncommutative two-oscillator algebra.* The generators have the following coordinate space representation on $L^2(\mathbb{R}^2, d\mathbf{x})$.

$$\begin{aligned}
\tilde{Q}_1 &= x + \frac{i\vartheta}{2}\frac{\partial}{\partial y} \qquad & \tilde{Q}_2 &= y - \frac{i\vartheta}{2}\frac{\partial}{\partial x} \\
\tilde{P}_1 &= -i\hbar\frac{\partial}{\partial x} \qquad & \tilde{P}_2 &= -i\hbar\frac{\partial}{\partial y}.
\end{aligned} \tag{8}$$

The free Hamiltonian and angular momentum operators are then:

$$\begin{aligned}
\tilde{H} &= -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}), \\
\tilde{M} &= -i\hbar\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}\right).
\end{aligned} \tag{9}$$

At this point, we define $\tilde{N}_i = m\tilde{Q}_i, \; i = 1, 2$, the resulting set of seven operators on $L^2(\mathbb{R}^2, d\mathbf{x})$ are easily found to satisfy the following set of commutation relations:

$$\begin{aligned} [\tilde{M}, \tilde{N}_i] &= i\hbar\epsilon_{ij}\tilde{N}_j & [\tilde{M}, \tilde{P}_i] &= i\hbar\epsilon_{ij}\tilde{P}_j \\ [\tilde{H}, \tilde{P}_i] &= 0 & [\tilde{M}, \tilde{H}] &= 0 \\ [\tilde{N}_i, \tilde{N}_j] &= i\epsilon_{ij}m^2\vartheta\tilde{I} & [\tilde{P}_i, \tilde{P}_j] &= 0 \\ [\tilde{N}_i, \tilde{P}_j] &= i\hbar\delta_{ij}m\tilde{I} & [\tilde{N}_i, \tilde{H}] &= i\hbar\tilde{P}_i, \end{aligned} \tag{10}$$

Now, if we choose $\hbar = 1$ and $\vartheta = \frac{\lambda}{m^2}$ in (10), we recover the same set of commutation relations as enumerated in (7). It is in this sense that the centrally extended (2+1) Galilei group can be considered as the symmetry group of *noncommutative quantum mechanics*, which is governed by the commutation relations (1).

A few remarks are in order. If we denote by $\hat{q}_i$ and $\hat{p}_i$, the standard quantum mechanical position and momentum operators, respectively, in a 2-plane, then the passage between the standard quantum mechanics and its noncommutative counterpart is demonstrated by the following linear, invertible and non-canonical transformations:

$$\begin{aligned} \tilde{Q}_1 &= \hat{q}_1 - \frac{\vartheta}{2\hbar}\hat{p}_2, \\ \tilde{Q}_2 &= \hat{q}_2 + \frac{\vartheta}{2\hbar}\hat{p}_1. \end{aligned} \tag{11}$$

Since, in (11), we obtain $\tilde{Q}_i = Q_i \Leftrightarrow \vartheta = 0$, the noncommutativity between the spatial degrees of freedom gets lost when the small parameter ϑ is turned off. However, by looking at equation (11), one captures the view of noncommutative quantum mechanics at the representation level only. The two distinct central generators of $G^{\text{ext}}_{\text{Gal}}$ are both mapped to the identity operator of $L^2(\mathbb{R}^2, d\mathbf{x})$. That being stated, the noncommutativity between the position coordinates seems to have been achieved by just taking a suitable linear combination of the standard quantum mechanical position and momentum operators and then declaring the newly obtained operators as their noncommutative counterparts! This is, indeed, true. But not the whole truth. The Lie group governing the symmetry of the noncommutative quantum mechanics in the 2-plane has been shown to be the centrally extended (2+1)-Galilei group unlike the 2-dimensional Weyl-Heisenberg group as in the case of standard quantum mechanics.

3. Coherent state Quantization related to a system with 2-spatial degrees of freedom

In this section, we first write down the unitary irreducible representations of $G^{\text{ext}}_{\text{Gal}}$. Then we compute the relevant coherent states arising from those UIRs and finally quantize the relevant phase space variables.

3.1 UIRs of $G^{\text{ext}}_{\text{Gal}}$

The unitary irreducible representations $\hat{U}_{m,\lambda}$ of $G^{\text{ext}}_{\text{Gal}}$ for nonzero values of both m and λ are computed in (Bose (1995b)). They are as follows

$$
\begin{aligned}
&(\hat{U}_{m,\lambda}(\theta,\phi,R,b,\mathbf{v},\mathbf{a})\hat{f})(\underline{\mathrm{k}}) \\
&\quad = e^{i(\theta+\phi)}e^{i[\mathbf{a}\cdot(\mathbf{k}-\frac{1}{2}m\mathbf{v})+\frac{b}{2m}\mathbf{k}\cdot\mathbf{k}+\frac{\lambda}{2m}\mathbf{v}\wedge\mathbf{k}]}s(R)\hat{f}(R^{-1}(\mathbf{k}-m\mathbf{v})),
\end{aligned} \tag{12}
$$

with $\hat{f}\in L^2(\hat{\mathbb{R}}^2, d\underline{\mathrm{k}})$. The UIRs, given by (12), are momentum space representations. The corresponding configuration space representations follow from the following Lemma.

Lemma 3.1. *The unitary irreducible representations of G^{ext}_{Gal} in the (two-dimensional) configuration space are given by*

$$
\begin{aligned}
&(U_{m,\lambda}(\theta,\phi,R,b,\mathbf{v},\mathbf{a})f)(\mathbf{x}) \\
&\quad = e^{i(\theta+\phi)}e^{im(\mathbf{x}+\frac{1}{2}\mathbf{a})\cdot\mathbf{v}}e^{-i\frac{b}{2m}\nabla^2}s(R)f\left(R^{-1}\left(\mathbf{x}+\mathbf{a}-\frac{\lambda}{2m}J\mathbf{v}\right)\right),
\end{aligned} \tag{13}
$$

where $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$, *$J$ is the 2×2 skew-symmetric matrix,* $J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, *and $f\in L^2(\mathbb{R}^2, d\mathbf{x})$.*

3.2 Coherent states of $G^{\text{ext}}_{\text{Gal}}$

It is evident from (13) that $U_{m,\lambda}$ is not square integrable, meaning that there exists no vector $\eta\in L^2(\mathbb{R}^2, d\mathbf{x})$ such that $\langle\eta, U_{m,\lambda}(g)\eta\rangle$ is finite in L^2-norm for all $g\in G^{\text{ext}}_{\text{Gal}}$.

Note that $H:=\Theta\times\Phi\times SO(2)\times\mathcal{T}$, having a generic element (θ,ϕ,R,b), is an abelian subgroup of $G^{\text{ext}}_{\text{Gal}}$. It immediately follows that the left coset space $X=G^{\text{ext}}_{\text{Gal}}/H$ is homeomorphic to $\mathbb{R}^4$ in view of the group law (6). Writing $\mathbf{q}$ for $\mathbf{a}$ and replacing $\mathbf{v}$ by $\mathbf{p}:=m\mathbf{v}$, the homogenous space X gets identified with the phase space of system of mass m moving in a two dimensional plane. We denote an element of X with $(\mathbf{q}, \mathbf{p})$. X carries an invariant measure $d\mathbf{q}\, d\mathbf{p}$ in $\mathbb{R}^4$ under the natural action of $G^{\text{ext}}_{\text{Gal}}$ on itself.

Let us now consider the following section $\beta : X \longmapsto G^{\text{ext}}_{\text{Gal}}$,

$$\beta(\mathbf{q}, \mathbf{p}) = (0, 0, \mathbb{I}_2, 0, \frac{\mathbf{p}}{m}, \mathbf{q}). \tag{14}$$

Now let $\chi \in L^2(\mathbb{R}^2, d\mathbf{x})$ is a fixed vector. At a later stage, we need to put some symmetry condition on it, but we leave it arbitrary at the moment. For each point $(\mathbf{q}, \mathbf{p}) \in X$, we define the vector $\chi_{\mathbf{q},\mathbf{p}} \in L^2(\mathbb{R}^2, d\mathbf{x})$ in the following way:

$$\chi_{\mathbf{q},\mathbf{p}} = U_{m,\lambda}(\beta(\mathbf{q}, \mathbf{p}))\chi\ . \tag{15}$$

Then using (13) and (14), one gets

$$\chi_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = e^{i(\mathbf{x}+\frac{1}{2}\mathbf{q})\ \cdot\mathbf{p}}\chi\left(\mathbf{x} + \mathbf{q} - \frac{\lambda}{2m^2}J\mathbf{p}\right). \tag{16}$$

At this point, we have the following lemma which we state without the proof.

Lemma 3.2. *For all $f, g \in L^2(\mathbb{R}^2)$, the vectors $\chi_{\mathbf{q},\mathbf{p}}$ satisfy the square integrability condition*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \langle f \mid \chi_{\mathbf{q},\mathbf{p}}\rangle\langle\chi_{\mathbf{q},\mathbf{p}} \mid g\rangle\ d\mathbf{q}\ d\mathbf{p} = (2\pi)^2\|\chi\|^2\langle f \mid g\rangle. \tag{17}$$

One can also show that the operator integral

$$T = \int_{\mathbb{R}^2\times\mathbb{R}^2} |\chi_{\mathbf{q},\mathbf{p}}\rangle\langle\chi_{\mathbf{q},\mathbf{p}}|\ d\mathbf{q}\ d\mathbf{p}\ ,$$

converges weakly to $T = 2\pi\ \|\chi\|^2\ I$. Let us now define the vectors

$$\eta = \frac{1}{\sqrt{2\pi}\|\chi\|}\chi, \quad \text{and} \quad \eta_{\mathbf{q},\mathbf{p}} = U(\beta(\mathbf{q}, \mathbf{p}))\eta\ , \quad (\mathbf{q}, \mathbf{p}) \in X\ . \tag{18}$$

The above set up, therefore, leads us to the following theorem.

Theorem 3.1. *The representation $U_{m,\lambda}$ in (13), of the extended Galilei group G^{ext}_{Gal}, is square integrable mod (β, H) and the vectors $\eta_{\mathbf{q},\mathbf{p}}$ in (18) form a set of coherent states defined on the homogeneous space $X = G^{ext}_{Gal}/H$, satisfying the resolution of the identity*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}|\ d\mathbf{q}\ d\mathbf{p} = I\ , \tag{19}$$

on $L^2(\mathbb{R}^2, d\mathbf{x})$.

Note that

$$\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = e^{i(\mathbf{x}+\frac{1}{2}\mathbf{q})\cdot\mathbf{p}}\,\eta\left(\mathbf{x}+\mathbf{q}-\frac{\lambda}{2m^2}J\mathbf{p}\right)\,. \tag{20}$$

We shall refer to the vectors $\eta_{\mathbf{q},\mathbf{p}} \in L^2(\mathbb{R}^2, d\mathbf{x})$ in (20), for all $(\mathbf{q},\mathbf{p}) \in X$ as the coherent states of Noncommutative quantum mechanics. It follows from (20) that in the limit $\frac{\lambda}{m^2} \to 0$, one recovers the *canonical coherent states* of standard quantum mechanics, if η is chosen to be a gaussian wave function. Since this also corresponds to setting $\lambda = 0$, it is consistent with constructing the coherent states of the (2+1)-Galilei group with one central extension (using only the first of the two cocycles in (5), with mass parameter m).

We want to emphasize again that the coherent states (20) are rooted in the underlying symmetry group of noncommutative quantum mechanics.

3.3 Noncommutative plane arising from coherent state quantization of the underlying phase space variables

The identification of the homogenous space $X = G_{\text{Gal}}^{\text{ext}}/H$ with the underlying phase space has been mentioned on several occasions in the preceding sections. We are now all set to carry out a coherent state quantization of functions on this phase space using the general technique outlined in Ali and Engliš (2005). It will turn out that such a quantization of the phase space variables of position and momentum will lead precisely to the operators (8).

On that note, the quantized version $\hat{\mathcal{O}}_f$ of a sufficiently well behaved function $f(\mathbf{q},\mathbf{p})$ defined on the underlying phase space reads (see, for example, Ali and Engliš (2005))

$$\hat{\mathcal{O}}_f = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(\mathbf{q},\mathbf{p})|\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}|\; d\mathbf{q}\; d\mathbf{p} \tag{21}$$

provided this operator is well-defined (again the integral being weakly defined). The operators $\hat{\mathcal{O}}_f$ act on a $g \in L^2(\mathbb{R}^2, d\mathbf{x})$ in the following manner

$$(\hat{\mathcal{O}}_f g)(\mathbf{x}) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(\mathbf{q},\mathbf{p})\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x})\left[\int_{\mathbb{R}^2}\overline{\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}')}g(\mathbf{x}')d\mathbf{x}'\right]\; d\mathbf{q}\; d\mathbf{p}\,. \tag{22}$$

If we now take the phase space functions to be the respective position and momentum coordinates, viz. q_i and p_i, $i = 1, 2$, then the resulting quantized operators $\hat{\mathcal{O}}_{q_i}$ and $\hat{\mathcal{O}}_{p_i}$ can immediately be read off from the following theorem Chowdhury and Ali (2013):

Theorem 3.2. *Let η be a smooth function which satisfies the rotational invariance condition, $\eta(\mathbf{x}) = \eta(\|\mathbf{x}\|)$, for all $\mathbf{x} \in \mathbb{R}^2$. Then, the operators $\hat{\mathcal{O}}_{q_i}, \hat{\mathcal{O}}_{p_i}$, $i = 1, 2$, obtained by a quantization of the phase space functions q_i, p_i, $i = 1, 2$, using the coherent states (18) of the (2+1)-centrally extended Galilei group, G^{ext}_{Gal}, are given by*

$$(\hat{\mathcal{O}}_{q_1} g)(\mathbf{x}) = \left(x_1 + \frac{i\lambda}{2m^2}\frac{\partial}{\partial x_2}\right) g(\mathbf{x}) \qquad (\hat{\mathcal{O}}_{q_2} g)(\mathbf{x}) = \left(x_2 - \frac{i\lambda}{2m^2}\frac{\partial}{\partial x_1}\right) g(\mathbf{x})$$

$$(\hat{\mathcal{O}}_{p_1} g)(\mathbf{x}) = -i\frac{\partial}{\partial x_1} g(\mathbf{x}) \qquad (\hat{\mathcal{O}}_{p_2} g)(\mathbf{x}) = -i\frac{\partial}{\partial x_2} g(\mathbf{x}), \tag{23}$$

for $g \in L^2(\mathbb{R}^2, d\mathbf{x})$, in the domain of these operators.

In (23) if we make the substitution $\vartheta = \dfrac{\lambda}{m^2}$, we get the operators (8) and the commutation relations of non-commutative quantum mechanics (with $\hbar = 1$):

$$[\hat{\mathcal{O}}_{q_1}, \hat{\mathcal{O}}_{q_2}] = i\vartheta I, \qquad [\hat{\mathcal{O}}_{q_i}, \hat{\mathcal{O}}_{p_j}] = i\delta_{ij} I, \qquad [\hat{\mathcal{O}}_{p_i}, \hat{\mathcal{O}}_{p_j}] = 0. \tag{24}$$

It is noteworthy in this context that the first commutation relation, between $\hat{\mathcal{O}}_{q_1}$ and $\hat{\mathcal{O}}_{q_2}$ in (24) above, also implies that the two dimensional plane $\mathbb{R}^2$ becomes noncommutative as a result of quantization.

4. Double and triple central extensions in $\mathbb{R}^4$ and noncommutative quantum mechanics

If we take the abelian group of translations G_T in $\mathbb{R}^4$ and centrally extend it using inequivalent local exponents (see, for example, Bargmann (1954) for definitions and preliminaries), the resulting central extensions play pivotal roles in noncommutative quantum mechanics as we shall try to discuss briefly in this section.

Let us denote by $(\mathbf{q}, \mathbf{p})$ as a generic element of G_T that obeys the group composition rule given by

$$(\mathbf{q}, \mathbf{p})(\mathbf{q}', \mathbf{p}') = (\mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}'), \qquad \mathbf{q}, \mathbf{p} \in \mathbb{R}^2. \tag{25}$$

At the level of the Lie algebra, all the generators commute with each other. In order to arrive at quantum mechanics out of this abelian Lie group, and to go further to obtain noncommutative quantum mechanics, we need to centrally extend this group of translations by some other abelian group, say by $\mathbb{R}$.

A double central extension of G_T will reproduce the noncommutative commutation relations (1), while a triple central extension of the underlying abelian group will lead the extended group generators to satisfy the commutation relations enumerated in (2).

The relevant central extensions of G_T are executed using inequivalent local exponents that are enumerated in the following theorem:

Theorem 4.1. *The three real valued functions ξ, ξ' and ξ'' on $G_T \times G_T$ given by*

$$\begin{aligned} \xi((q_1,q_2,p_1,p_2),(q_1',q_2',p_1',p_2')) &= \frac{1}{2}[q_1p_1' + q_2p_2' - p_1q_1' - p_2q_2'], \quad (26)\\ \xi'((q_1,q_2,p_1,p_2),(q_1',q_2',p_1',p_2')) &= \frac{1}{2}[p_1p_2' - p_2p_1'], \quad (27)\\ \xi''((q_1,q_2,p_1,p_2),(q_1',q_2',p_1',p_2')) &= \frac{1}{2}[q_1q_2' - q_2q_1'], \quad (28)\end{aligned}$$

are inequivalent local exponents for the group, G_T, of translations in $\mathbb{R}^4$.

The group composition rule for the extended group $\overline{\overline{G_T}}$ reads

$$\begin{aligned}&(\theta,\phi,\mathbf{q},\mathbf{p})(\theta',\phi',\mathbf{q}',\mathbf{p}')\\ &\quad = (\theta+\theta'+\frac{\alpha}{2}[\langle\mathbf{q},\mathbf{p}'\rangle - \langle\mathbf{p},\mathbf{q}'\rangle], \phi+\phi'+\frac{\beta}{2}[\mathbf{p}\wedge\mathbf{p}'], \mathbf{q}+\mathbf{q}', \mathbf{p}+\mathbf{p}'), \quad (29)\end{aligned}$$

where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. Also, $\langle .,. \rangle$ and $\wedge$ are defined as $\langle \mathbf{q}, \mathbf{p}\rangle := q_1p_1 + q_2p_2$ and $\mathbf{q}\wedge\mathbf{p} := q_1p_2 - q_2p_1$ respectively.

A group element of the centrally extended group $\overline{\overline{G_T}}$, where the extension is achieved by means of ξ and ξ', has the matrix representation by the following 7×7 matrix

$$(\theta,\phi,\mathbf{q},\mathbf{p})_{\alpha,\beta} = \begin{bmatrix} 1 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta\\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi\\ 0 & 0 & 1 & 0 & 0 & 0 & q_1\\ 0 & 0 & 0 & 1 & 0 & 0 & q_2\\ 0 & 0 & 0 & 0 & 1 & 0 & p_1\\ 0 & 0 & 0 & 0 & 0 & 1 & p_2\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}. \quad (30)$$

Let us denote the generators of the Lie group $\overline{\overline{G_T}}$, or equivalently the basis of the associated Lie algebra, $\overline{\overline{\mathcal{G}_T}}$ by $\Theta, \Phi, Q_1, Q_2, P_1$ and P_2. They are found to satisfy the following set of commutation relations

$$\begin{aligned} &[P_i, Q_j] = \alpha\delta_{i,j}\Theta,\ [Q_1, Q_2] = \beta\Phi, \quad [P_1, P_2] = 0, \quad [P_i, \Theta] = 0, \\ &[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [\Theta, \Phi] = 0, \quad i, j = 1, 2\ . \end{aligned} \tag{31}$$

It is easily seen from (31) that Θ and Φ form the center of the algebra $\overline{\overline{\mathcal{G}_T}}$.

Also, unlike in standard quantum mechanics, the two generators of space translation, Q_1, Q_2, no longer commute, the noncommutativity of these two generators being controlled by the central extension parameter β. It is in this context reasonable to call the Lie group $\overline{\overline{G_T}}$ the *noncommutative Weyl-Heisenberg group* and the corresponding Lie algebra the *noncommutative Weyl-Heisenberg algebra.*

The Lie group $\overline{\overline{G_T}}$ is nilpotent as can be easily seen from its matrix representation (30). Hence, the powerful method of orbits (see Kirillov (2004)) can be applied to classify all its unitary irreducible representations. A unitary irreducible representation of $\overline{\overline{G_T}}$ is supplied by the following theorem:

Theorem 4.2. *The noncommutative Weyl-Heisenberg group* $\overline{\overline{G_T}}$ *admits a unitary irreducible representation realized on* $L^2(\mathbb{R}^2, d\mathbf{s})$ *by the operators* $U(\theta, \phi, \mathbf{q}, \mathbf{p})$*:*

$$(U(\theta, \phi, \mathbf{q}, \mathbf{p})f)(\mathbf{s}) = \exp i\left(\theta + \phi - \alpha\langle \mathbf{q}, \mathbf{s} + \frac{1}{2}\mathbf{p}\rangle - \frac{\beta}{2}\mathbf{p}\wedge\mathbf{s}\right) f(\mathbf{s} + \mathbf{p}), \tag{32}$$

where $f \in L^2(\mathbb{R}^2, d\mathbf{s})$.

In order to execute a triple central extension of G_T, we make use of all three inequivalent local exponents ξ, ξ', and ξ'' enumerated in theorem 4.1. The group composition rule for the centrally extended group $\overline{\overline{\overline{G_T}}}$ then reads

$$\begin{aligned} &(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})(\theta', \phi', \psi', \mathbf{q}', \mathbf{p}') \\ &\quad = (\theta + \theta' + \frac{\alpha}{2}[\langle \mathbf{q}, \mathbf{p}'\rangle - \langle \mathbf{p}, \mathbf{q}'\rangle], \phi + \phi' + \frac{\beta}{2}[\mathbf{p}\wedge\mathbf{p}'], \psi + \psi' + \frac{\gamma}{2}[\mathbf{q}\wedge\mathbf{q}'] \\ &\quad\ , \mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}'). \end{aligned} \tag{33}$$

The relevant matrix representation of $\overline{\overline{\overline{G_T}}}$ is given by the following 8×8 upper triangular matrix

$$(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})_{\alpha,\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & \frac{\gamma}{2}q_1 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (34)$$

Unlike the double extension scenario, we have three central generators for the triply extended group $\overline{\overline{\overline{G_T}}}$. The additional central generator Ψ is responsible for the noncommutativity in the momentum coordinates. Commutation relations between the respective generators now read

$$\begin{aligned} &[P_i, Q_j] = \alpha\delta_{i,j}\Theta, \quad [Q_1, Q_2] = \beta\Phi, \quad [P_1, P_2] = \gamma\Psi, \quad [P_i, \Theta] = 0, \\ &[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [P_i, \Psi] = 0, \\ &[Q_i, \Psi] = 0, \quad [\Theta, \Phi] = 0, \quad [\Phi, \Psi] = 0, \quad [\Theta, \Psi] = 0, \quad i, j = 1, 2\ . \end{aligned} \qquad (35)$$

A unitary irreducible representation of $\overline{\overline{\overline{G_T}}}$ is supplied by the following theorem:

Theorem 4.3. *The triply extended group of translations $\overline{\overline{\overline{G_T}}}$ admits a unitary irreducible representation realized on $L^2(\mathbb{R}^2)$. The explicit form of the representation is given by*

$$\begin{aligned} &(U(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r_1, s_2) \\ &\quad = e^{i(\theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2}q_1 p_1 - \frac{\alpha}{2}q_2 p_2)} e^{i(\phi - \beta p_1 s_2 - \frac{\beta}{2}p_1 p_2)} \\ &\quad \times e^{i(\psi + \gamma q_2 r_1 + \frac{\gamma}{2}q_2 q_1)} f(r_1 + q_1, s_2 + p_2), \end{aligned} \qquad (36)$$

where $f \in L^2(\mathbb{R}^2, dr_1 ds_2)$.

5. Conclusion and future perspective

In this paper, we have derived the commutation relations of noncommutative quantum mechanics (NCQM) by different means. We first discussed the role of the centrally extended (2+1)-Galilei group $G^{\text{ext}}_{\text{Gal}}$ as a symmetry group of NCQM from an algebraic point of view. Later we obtained a family of *noncanonical coherent states* from the UIRs of $G^{\text{ext}}_{\text{Gal}}$ and called them the coherent states of NCQM. Using these coherent states, we quantized the position and the momentum coordinates of a phase space related to a system of two degrees of freedom, the symmetry of which is dictated by $G_{\rm Gal}^{\rm ext}$. The ensuing quantized variables were found to satisfy the same set of commutation relations as those of NCQM. Then, we considered the double central extension of the abelian group of translations G_T in $\mathbb{R}^4$ whose generators were found to satisfy the commutation relations of NCQM. A triple central extension of G_T was executed finally to make the momentum coordinates noncommutative, as well. In Chowdhury and Ali (2014), we computed the unitary dual of the triply extended group $\overline{\overline{\overline{G_T}}}$ and discussed how various gauges used in NCQM are related to this unitary dual. In a later publication, we propose to compute the Wigner function associated with $\overline{\overline{\overline{G_T}}}$ and study aspects of Berezin-Toeplitz quantization pertaining to the noncommutative coherent states obtained here in this paper.

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